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A SIMPLE EDGE ELIMINATION CRITERION
SEARCH FOR THE CENTER OF A GRAPH

by

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A SIMPLE EDGE ELIMINATION CRITERION IN A SEARCH FOR THE CENTER OF A GRAPH

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ABSTRACT

The note proposes a new edge elimination criterion in an algorithm whose purpose is to locate the center of a graph. The new criterion requires the same amount of information and computation but generates a stronger lower bound. It therefore reduces substantially the number of edges that have to be thoroughly and costly checked for a possible center point along them.

For a graph $G=(V,E)$ with a set of vertices V and a set of undirected edges E the center function at point $x\varepsilon G$ is defined as $z_c(x)=\text{Max}\{d(x,v)/v\varepsilon V\}$, where $d(x,y)$ is the length of the shortest path connecting x and y in G. The center of the graph is the point which minimizes $z_c(x)$ over G. The major difficulty in locating the center of a graph is a result of the fact that, in general, the center function $z_c(x)$ may have local and hence global minima along an edge $e\varepsilon E$. This implies that for any given edge $e\varepsilon E$ a search procedure for the center should either identify and check local minima on e or, using some elimination criterion, discard e as an edge which is unable to generate the center point. Such elimination criterion usually depends on vertex information only, which is assumed to be known, and some given upper bound on the minimal value of the center function $z_c(x)$. For $e=(r,s)\varepsilon E$ with $r,s\varepsilon V$ let $b(r,s)$ denote the length of e, and $b(r,x)$ $\left(b(s,x)\right)$ be the distance, along e, between r(s) and $x\varepsilon e$. For any $x\varepsilon e$, a lower bound on $z_c(x)$ is $L_c(x)=\text{Max}\left\{z_c(r)-b(r,x),\ z_c(s)-b(s,x)\right\}$. Therefore, the best lower bound for $\text{Min}\{z_c(x)/x\varepsilon e\}$ which may be stated using only the information $z_c(r)$, $z_c(s)$ and $b(r,s)$ is

$$L_c(x^*)=\text{Min}\{L_c(x)/x\varepsilon e\}=\left(z_c(r)+z_c(s)-b(r,s)\right)/2$$

where $b(r,x^*)=\left(b(r,s)+z_c(r)-z_c(s)\right)/2$. See figure.

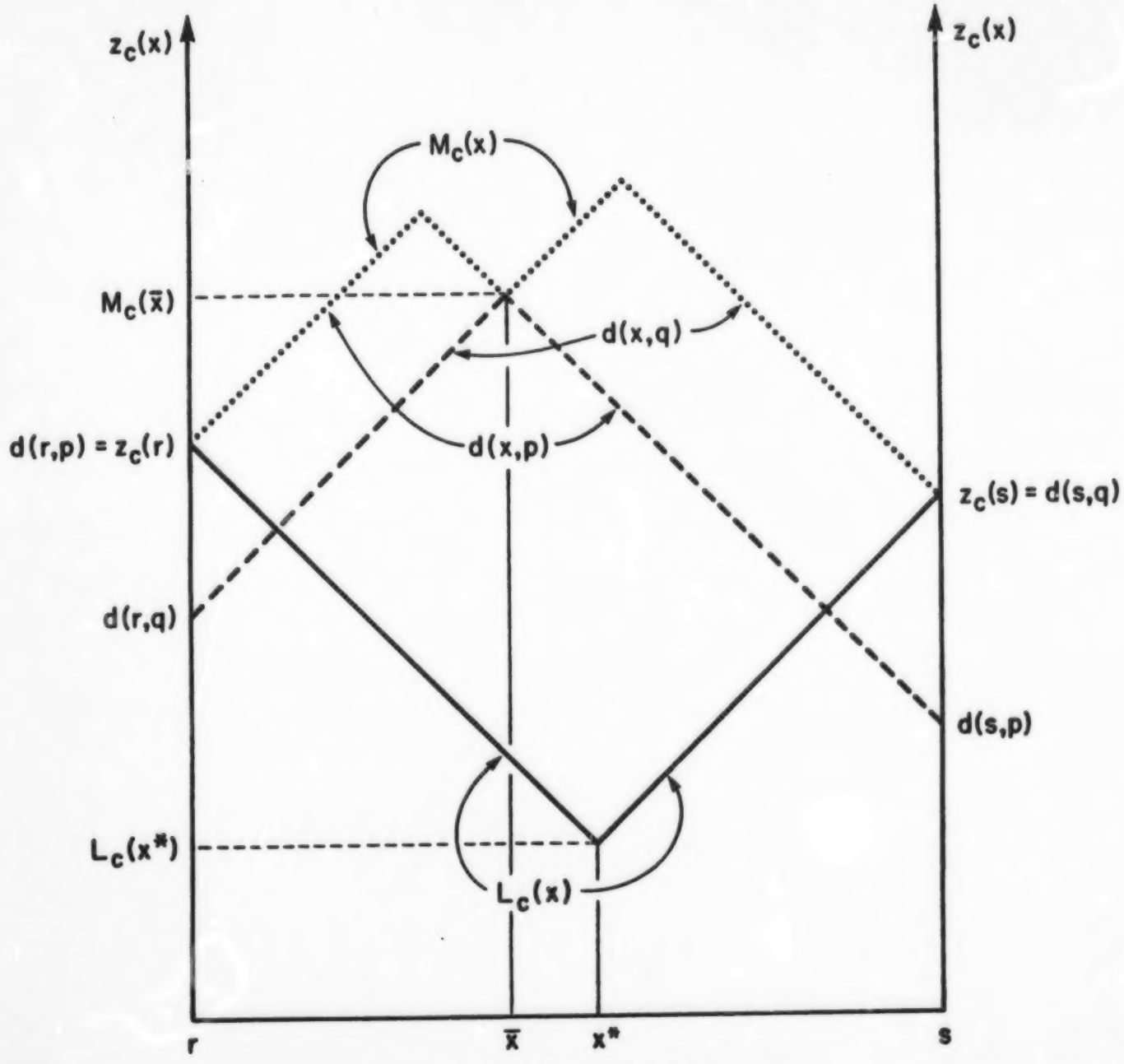
Given some upper bound, say $\bar{z}$, for $\text{Min}\{z_c(x)/x\varepsilon G\}$, then $L_c(x^*)\geq\bar{z}$ implies that no point $x\varepsilon e$ may be the center of G and the edge e may be discarded without actually constructing the function $z_c(x)$ for $x\varepsilon e$. This simple elimination criterion was used, for example, by Handler [3] and Odoni [4] and a similar one by Christofides [1], in their studies of the centers of graphs, and by Halpern [2] in his construction of the cent-dian function. The usefulness of such simple elimination criterion depends on its efficiency, namely, how many edges does it successfully eliminate. The purpose of this note is to propose a stronger lower bound which requires the same amount of information and an equal computational effort.

For $e=(r,s)$ assume that $z_c(r)=d(r,p)$ $\left(z_c(s)=d(s,q)\right)$, anmely the farthest vertex from r(s) is p(q). The information which is used for the new lower bound is $d(r,q)$ and $d(s,p)$ instead of $z_c(r)$ and $z_c(s)$. By definition $z_c(r)\geq d(r,q)$ and

$z_c(s) \geq d(s,p)$. Clearly, $M_c(x)=\text{Max}\{d(x,p), d(x,q)\}$ is a lower bound on $z_c(x)$ for any $x \in e$. Within the edge e (i.e., the vertices r and s excluded) the minimal value of $M_c(x)$ is $M_c(\bar{x})=\left(b(r,s)+d(s,p)+d(r,q)\right)/2$ and it is attained at point $\bar{x} \in e$ for which $b(r,\bar{x})=\left(b(r,s)+d(s,p)-d(r,q)\right)/2$ (see figure). Thus, the given upper bound $\bar{z}$ is now compared with $M_c(\bar{x})$ and if $M_c(\bar{x}) \geq \bar{z}$ then the edge e is discarded as a potential generator of the center of G. It is simple to show that $M_c(\bar{x}) \geq L_c(x^*)$ with equality holds only if both d(x,p) and d(x,q) are linear over all of e.

A lower bound for $\text{Min}\{z_c(x)/x \in e\}$ is used in an effort to eliminate edge e, in Halpern's algorithm (step 4, p. 543 [2]) for the construction of the cent-dian function. The two lower bounds $L_c(x^*)$ and $M_c(\bar{x})$ were compared for a large number of test graphs. For each graph the number of edges successfully eliminated by each of the two bounds was recorded. The table provides the average (of ten different graphs of the same size) percentage of edges (out of the total number of edges, $|E|$) that were eliminated by the two criteria. For the sizes reported in the table, as well as for many others, M_c performed significantly and uniformly better than L_c.

$\|V\|$, $\|E\|$	Percentage of edges eliminated	
	$M_c(\bar{x})$	$L_c(x^*)$
35,350	96.1	17.8
45,350	95.9	22.5
55,350	98.5	39.8
35,150	96.9	48.1
35,550	95.1	7.5


$z_c(x)$
$z_c(x)$
$M_c(x)$
$M_c(\bar{x})$
$d(x,q)$
$d(x,p)$
$d(r,p) = z_c(r)$
$z_c(s) = d(s,q)$
$d(r,q)$
$d(s,p)$
$L_c(x^*)$
$L_c(x)$
r
$\bar{x}$
x^*
s

edge e = (r, s)

REFERENCES

[1] Christofides, N., *Graph Theory: An Algorithmic Approach*, Academic Press, 1975.

[2] Halpern, J., "Finding Minimal Center-Median Convex Combination (Cent-Dian) of a Graph" Management Science, Vol. 24 (1978), pp. 535-544.

[3] Handler, G. Y., "Minimax Network Location: Theory and Algorithms", Ph.D. Thesis. Technical Report No. 107, Operations Research Center, M.I.T., Cambridge, Massachusetts, (November 1974).

[4] Odoni, A. R., "Location of Facilities on a Network: A Survey of Results", Technical Report TR-03-74, Operations Research Center, M.I.T., Cambridge, Massachusetts (April 1974).